MATH 105A and 110A Review: Eigenvalues and diagonalization

Facts to Know:

Let A be any $n \times n$ matrix. A nonzero vector x is said to be an **eigenvector** of A if

$Ax = \lambda x$ for some number λ

The λ above is called an eigenvalue.

λ is an eigenvalue of A if and only if

$$(A - \lambda \cdot I)x = 0$$

has a nontrivial solution. We find eigenvalues by solving the **characteristic polynomial**:

$$0 = \det(A - \lambda \cdot I)$$

We say $\lambda = a$ is an eigenvalue of A with **multiplicity** k if

$$0 = \det(A - \lambda \cdot I) = (a - \lambda)^k s(\lambda),$$

where $s(a) \neq 0$.

A matrix D is said to be a **diagonal** matrix if:

$$D = \begin{bmatrix} * & & 0 \\ & * \ddots & \\ 0 & & * \;* \end{bmatrix} \qquad 2\times 2: \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

A matrix A is said to be **diagonalizable** if there exists a diagonal matrix D and some invertible matrix P such that

$$A = PDP^{-1}$$

Any $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

The diagonalization of A:

eigenvectors $x_1, x_2, \dots, x_n$

eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}^{-1}}_{P^{-1}}$$

Any $n \times n$ matrix A is diagonalizable if and only if the following two hold:

1. All the eigenvalues are real.
2. If λ is an eigenvalue of multiplicity k, then there are k linearly ind. eigenvectors corresponding to λ.

Examples:

1. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

Step1 Find eigenvalues and the mult.

$$0 = \det(A - \lambda \cdot I) = \det\begin{pmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{pmatrix}$$

$$= (7-\lambda)(1-\lambda) + 8$$

$$= 7 - 7\lambda - \lambda + \lambda^2 + 8$$

$$= \lambda^2 - 8\lambda + 15$$

$$= (\lambda - 3)(\lambda - 5)$$

$\lambda = 3$ with mult. 1

$\lambda = 5$ with mult. 1

Step2 Find eigenvectors:

$\lambda = 3$ solve

$$(A - 3I)x = 0$$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$$

$$4x_1 + 2x_2 = 0$$

$$\Rightarrow x_2 = -2x_1$$

$$x = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The eigenvector is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$\underline{\lambda = 5}$

$$\begin{bmatrix} 2 & 2 \\ -4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$x = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\underline{\text{step 3:}}$

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} = (*)$$

$$A \quad = \quad P \quad D \quad P^{-1}$$

$$(*) = \begin{bmatrix} 3 & -5 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$